Probability Final

You may use your sheet from the midterm and a new sheet of notes. No calculators, cell phones, PDA’s, laptops, or HAL 9000’s. Show your reasoning. Don’t just give the answer.

1. A box contains two gold balls and two clay balls. You are allowed to choose successively balls from the box at random. You win 1 dollar each time you draw a gold ball and lose 1 dollar each time you draw a clay ball. After a draw, the ball is not replaced.

   (a) If you draw exactly one ball, what is your expected earnings?

   (b) What is the moment generating function for the value of the first draw?

   (c) If you draw exactly $k$ balls (for $k = 1, 2, 3, 4$) what is your expected earnings?

   (d) If you draw until you are ahead by 1 dollar or until there are no more gold balls, what is your expected earnings?

2. Suppose you win 1 dollars when an black card is drawn from a deck of cards but you lose 1 dollar when a red card is drawn. (So out of the 52 cars, you win with 26 of them and lose with 26 of them.)

   (a) Let $X_1$ be the amount you win on the first draw, and $X_2$ be the amount you win on the second draw. (Assume you don’t put the card back.) What is Cov($X_1, X_2$)?

   (b) What is the mean and variance of $X_1 + X_2$?

   (c) What is the mean and variance of $\sum_{i=1}^{52} X_i$? (Hint: think before you compute.)

3. Consider a non-negative random variable: $X \geq 0$.

   (a) If $E(X) = 1$, what is a good bound on $P(X \geq 100)$?

   (b) If $E(X) = 1$, and $V(X) = 1$ what is a good bound for $P(X \geq 100)$?

   (c) If the generating function $h_X(2) = 4$, (i.e. $E(2^X) = 4$ then what is a good bound for $P(X \geq 100)$?

4. Suppose the moment generating function for $X$ is $g(t) = 1 + t$. In other words, $E(e^{tX}) = e^t$. What can you tell me about $X$?

5. The law of large numbers tells us alot about a sum of random variables. The CLT tells us even more about sums. But what about products? Let $X_i$ be a random variable that takes on either $+1$ or $-1$ with equal probability. Let $P_n = \prod_{i=1}^n X_i$. Will $P_n$ converge to some fixed value? (I.e. law of large numbers?) If it converges, what is this value, if it doesn’t converge, what does $P_n$ look like?

6. Statistics is often driven by two things, a prediction and a residual. Define the random variable $Z = E(Y|X)$ and the random variable $W = Y - Z$. Then $Z$ is the prediction and $W$ is the residual of the “regression” of $Y$ on $X$. 
(a) What is $E(Z)$?
(b) What is $E(XZ)$
(c) Let $h()$ be an arbitrary function, show $E(h(X)Z) = E(h(X)Y)$.
(d) What is $E(WZ)$?

7. Let $X_i$ be a random variable with mean 1.01 and standard deviation .2. (For example, $X = 1.21$ or $X = .81$ with equal probability, but that is such an ugly statement, let’s pretend I didn’t mention it.) Let $W = \prod_{i=1}^{n} X_i$. Suppose all the $X_i$’s are independent, so the whole series is IID.

(a) What is $E(W)$?
(b) What is the long run growth rate (i.e. $\lim_{n \to \infty} (\log W_n)/n$)?
(c) What will $W_{1000}$ look like?

8. Suppose you put 100 mice on a calorie restriction diet. Normal mice on a normal diet live 1000 days with a standard deviation of 150 days.

(a) If this diet doesn’t change the length of life for these mice, what will be the mean, variance and distribution of $T$? (Where $T$ is the average number of days a mouse in the experiment lives.)

(b) Find a good estimate the probability that $T$ is bigger than 1300.

(c) If your experiment actually yielded an average of 1300, would you believe that these mice have the same mean as typical mice?

(d) (bonus) Using generating functions, provide a better bound.