Chapter 8 intro

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Administrivia

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Markov’s inequality: why

Definition of expectation comes from probability.
How do you compute a probability from an expectation?

• working backwards—like division

• Expectation is linear, probability is linear, just “invert” the matrix that relates them.

• harder, but more useful, find a bound

Markov’s inequality: motivation

Let’s ask Gretchen: Suppose I have 10 kids, and the average number of blocks per kid is 6.5. What maximal amount any one kid has? (65)
What is the probability of a kid choosen at random to have more than 65 blocks? (zero) Ok, how about 64 blocks?

**Imprtant to do with blocks and not money**

The average wealth saved by my 10 friends is $10,000. What is the maximum? $200k I say? Ah, several of my friends are in debt hence negative savings.

**Statement and proofs:**

If, $X \geq 0$, then $P(X > M) < E(X)/M$.

Proofs:

- Draw balance beam and optimize
- Consider the random variable $Y = MI_{X>M}$. $Y \leq X$, so $EY < EX$.
- $EX = \int_0^\infty xf(x)dx \geq \int_M^\infty xf(x)dx \geq \int_M^\infty Mf(x)dx = P(X > M)$.

**Chebeychev’s inequality**

Let $X = (Y - E(Y))^2$. Then $X \geq 0$ so markov applies.

**Theorem 1** $P(|Y - E(Y)|^2 > M) \leq E(Y - E(Y))^2/M$

Even that theorem wouldn’t get a new name. But lets do a substitution of $k = \sqrt{M}$.

**Theorem 2 (Chebeychev)** $P(|Y - E(Y)| > k) \leq Var(Y)/k^2$
Exponential inequality

Let \( X = e^Y \). Then \( X \geq 0 \), so Markov applies.

**Theorem 3** Let \( M(1) = E(e^Y) \), then

\[
P(Y > k) \leq M(1)e^{-k}
\]

- \( M(s) = E(e^{sY}) \). So new theorem

\[
P(Y > k) \leq M(s)e^{-sk}
\]

Reminder about sums of random variables

Let \( X_i \) be a sequence of IID random variables. Let \( S = \sum_{i=1}^{n} X_i \). Then:

\[
E(S) = nE(X)
\]
\[
Var(S) = nVar(X)
\]
\[
M_S(1) = M_X(1)^n
\]

If the RHS’s exist.

**Application**

Plug in the blanks:

\[
P(S > nk) \leq E(X)/k \quad \text{if } X \geq 0
\]
\[
P(|S - n\mu| > k) \leq nVar(X)/k^2
\]
\[
P(S > k) \leq M_X(1)^{-k/n}
\]

But that is too easy. Let’s build it up by hand.
Gambling

- The setup:
  - Suppose round $i$ you bet $\epsilon$ fraction of your wealth on gamble $Y_i$.
  - If it is a fair bet, then your gain is $\epsilon WE(Y)$ (which is a random variable!) or just zero.
  - Suppose your initial wealth is 1.
  - What is your expected wealth at time $T$? Also 1.

- So, by markov $P(W > k) \leq 1/k$.

- Converting to sums:
  - But if your bet either pays out or doesn’t pay out.
  - Let $S$ be the number of times it pays out.
  - Then $W = (1 + \epsilon a)^S (1 - \epsilon b)^{n-S}$.

- Plugging in:
  \[
P(W > k) = P( (1 + \epsilon a)^S (1 - \epsilon b)^{n-S} > k ) \\
  = P( (\frac{1 + \epsilon a}{1 - \epsilon})^S (1 - \epsilon b)^n > k ) \\
  = P( \alpha^S > k(1 - \epsilon b)^{-n} ) \\
  = P( S \log(\alpha) > \log(k(1 - \epsilon b)^{-n}) ) \\
  = P( S > \log(k(1 - \epsilon b)^{-n})/\log(\alpha) ) \\
  \leq 1/k
  \]

Writing this differently, let
  \[
v = \log(k(1 - \epsilon b)^{-n})/\log(\alpha)
  \]
\[
\log(\alpha)v = \log(k(1 - \epsilon b)^{-n}) \\
\alpha^v = k(1 - \epsilon b)^{-n} \\
(1 - \epsilon b)^n \alpha^v = k
\]

So,

\[
P(S > v) \leq \alpha^{-v}/(1 - \epsilon b)^n
\]