Chapter 7.1 teaching notes

Dean P Foster

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Administrivia

- Homework questions?

Many faces of conditional expectation

There are many ways to define conditional expectation:

- \( E(Y \mid X = x) = \sum_x x P(Y = x \mid X = x) \) (or \( E(Y \mid X = x) = \int_x x f(Y = x \mid X = x) dx \))

- \( E(Y \mid X) = g(X) \) such that \( g(x) = E(Y \mid X = x) \)

- \( E(Y \mid X) = g(X) \) such that \( E(g(X)H(X)) = E(Y H(X)) \)

- Fair price: (small bets, repeated many times, actions)

- Best forecast: \( E(Y \mid X) = g(X) \) such that for all \( h() \) we have \( E(Y - h(X))^2 \geq E(Y - g(X))^2 \)

- Covariance gives us a new one.
Define $E(\cdot | X) = X \text{cov}(X, X)^{-1} \text{cov}(X, \cdot)$

- Technical condition: $AX = X^2$ for some matrix $A$.
  - example $X = \{0, 1\}$
  - example $X = [Z, Z^2, Z^3, \ldots, Z^k]$ where $k$ is the number of discrete values $Z$ takes on

- In statistics, called regression

- For homework you checked that $EX \text{cov}(X, X)^{-1} \text{cov}(X, Y) = E(Y)$.

- Check that it works:
  \[
  E(Y|X) = X \text{cov}(X, X)^{-1} \text{cov}(X, Y)
  \]
  \[
  g(x) = x \text{cov}(X, X)^{-1} \text{cov}(X, Y)
  \]
  \[
  E(g(X)h(X)) = \sum h_i E(g(X)X^i)
  \]
  \[
  = \sum h_i E(X \text{cov}(X, X)^{-1} \text{cov}(X, Y)X^i)
  \]
  \[
  = \sum h_i E(X^{i+1} \text{cov}(X, X)^{-1} \text{cov}(X, Y))
  \]
  \[
  = \sum h_i E(A^i \text{cov}(X, X)^{-1} \text{cov}(X, Y))
  \]
  \[
  = \sum h_i A^i E(Y)
  \]

YIKES!!!

**Sums of discrete random variables**

Consider $X, Y$ independent discrete random variables. Let $Z = X + Y$. What is the distribution of $Z$?
\[ P(Z = z) = \sum_{x+y=z} P(X = x, Y = y) \]
\[ = \sum_x P(X = x, Y = z - x) \]
\[ = \sum_x P(X = x)P(Y = z - x) \]

Pretty for integer valued random variables. Define \( m_x, m_y \) and \( m_z \) as the distributions then:

\[ m_z(j) = \sum_k m_x(k)m_y(z - x) \]

Curious fact: Sums of any finite integer valued random look the same.

**Sums of continuous random variables**

same idea:

\[ (f \ast g)(z) = \int f(z - y)g(y)dy = \int f(x)g(z - y)dx \]

- uniform goes to triangle
- exponential is a gamma
- gamma goes to gamma: parameters add. Now we have an excuse as to why gamma is defined as it is: \( \Gamma_n(x) = (\Gamma_a \ast \Gamma_b)(x) \).
- normal goes to normal: means add, variances add
• Cauchy goes to Cauchy \((1/\pi(1 + x^2))\)

• Average of Cauchy goes to SAME Cauchy