Chapter 6 teaching notes

Dean P Foster

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Administrivia

• Hand back homeworks and midterm

(1/2) factorial

• Extending the factorial
• The gamma function: $\Gamma(n + 1) = n!$
• $\Gamma(n + 1) = n! = \int_0^\infty t^n e^{-t} dt$
• Plot: $\Gamma(0) = +\infty$ and $\Gamma(-1) = -\infty$
• Relationship to volume of a sphere and hence a normal distribution

Expectation

• Easy definition $E(X) = \sum_x xP(X = x)$
– Oops! Not all sums are defined
– Call the undefined ones as “undefined.”

• Example: $X =$ number flips until a head. What is $E(X)$?
• Example: $Y = 2^X$.
  – Is $E(Y) = 2^{E(X)}$? No!
  – Compute.
  – Oops. Makes sense to call it $+\infty$ rather than undefined. But tradition precludes us from doing so.
  – What about $Y = 1.9999^X$ and $Y = 2.0001^X$?
  – For shadow: $Y = 1.0001^X$ is also easy and useful

**Interpretation**

• Naive: “Fair price” (Good for $X$ but not for $Y$)
• Long run average. (Good for intuition–but not for philosophy)
• Typical value. (Again good for $X$ but not for $Y$)
• Gradually it will come to have meaning–but don’t lean on the meaning too hard. This is mathematics!

**Sums of random variables**

• $A =$ first die, $B =$ second die, Then $E(A) = E(B) = 3.5$.
• What about $E(A + B)$? Clearly 7! But mathematically why?
• Theorem: $E(A + B) = E(A) + E(B)$ and $E(cA) = cE(A)$.

• Prove it.

**Application**

• Consider a random permutation?

• What does this mean? Card shuffling!

• How many fixed points?

• Technical definition: $\sum x m(x)$ requires inclusion / exclusion and lots of pretty math.

• What is the expected number of fixed points in the first position? Easy, $1/n$.

• Now add them up!

• I love this book!

**Debriefing**

• Three key ideas:
  
  – Linearity of expectation
  
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• Ok, how about:
  
  – Linearity of expectation
Indicator variables
”Fundamental mystery of probability”

Closing: Linearity of expectation rules

- Alternative definitions of expectation:
  - Ours: $E(X) = \sum x m(x)$
  - Integration 1: $E(X) = \int x f(x) dx = \text{limit of Riemann sums}$
  - Integration 1: $E(X) = \int x dF(x)$
  - Measure theory: $E(X) = \lim \sum x P(x < X < x + \epsilon) = \text{Lebesgue integral}$

- All satisfy fundamental mystery of probability
- All map the space of random variables to the real line
  - Think of all possible (nice) random variables
  - We can put them in a vector space
  - Cool! it is infinite dimensional
  - Expectation then is a linear operator on this space
  - Can be thought of then as an inner product: $\langle 1, X \rangle$