Chapter 4.2 teaching notes

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Administrivia

- Exam 1 weeks from today

Continuous conditional probability

Table form of random variables:

- Do Age (in decades) vs weight (in stones)
  - Question: What is average age for 40 year old? (About 13)
- Redo Age (in years) vs weight (in kg)
- Redo Age (in days) vs weight (in g)
- Redo Age (in nanosec) vs weight (in picograms)
- Limit is continuous

Usual definition: call numbers in cells $f(x, y)$. Call sums $f_X(x)$ and $f_Y(y)$. Now conditional probability is $P(X = x|Y = y) = f(x, y)/f_Y(y)$. But this is morally wrong! The book avoids this definition.
Book’s view:

\[ f(x|E) = f(x)/P(E), \text{ if } x \in E \]

and zero otherwise.

- No divide by zero, so no immorality!
- No need for limits or calculus

Exponential distribution

How long for an atom to decay?

- Density is \( f(x) = \lambda e^{-\lambda x} \)
  - Picture
  - meaning of lambda
  - typical = 1/lambda
  - “half life” = .69/lambda

- \( E = \{X > r\} \) and \( F \equiv \{X > r + s\} \) what is \( P(F|E) \)?
- Define a new random variable \( Y = (X - r) \lor 0 \).
- \( f_Y(y|E) = f(x + r)/P(E) = f(x) \) if \( E \) occurs.

Independent random variables

- table form: \( P(X = x, Y = y) = P(X = x)P(Y = y) \)
• Evil density form: $f(x, y) = f_X(x)f_Y(y)$ (Why evil? Left hand side is only meaningful when integrated with $d[x, y]$. RHS only when sequentially integrated. So we need Fatou.)

• Wonderful probability form: $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$

• In fact, only need very simple $A$’s and $B$’s.

**Definition of independence**

$X_1, X_2, \ldots$ are independent if $F(x_1, x_2, \ldots) = \prod F(x_i)$

**Tirade**

Most books go off on double integrals at this point. The key thing is, can you do the integral. who cares if you get a negative probability—you did the arc-tan substitution correctly. We care! Since we are doing it correctly, we can do interesting things—like the beta-binomial.

**Beta distribution**

$$B(\alpha, \beta, x) = \begin{cases} 
(1/B(\alpha, \beta))x^{\alpha-1}(1-x)^{\beta-1}, & \text{if } 0 \leq x \leq 1, \\
0, & \text{otherwise.}
\end{cases}$$

First definition of $B$:

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \, dx.$$  

Second definition of $B$:

$$B(\alpha, \beta) = \frac{(\alpha - 1)!(\beta - 1)!}{(\alpha + \beta - 1)!}.$$
So,

\[
\left( \frac{\alpha - 1 + \beta - 1}{\alpha - 1} \right) = \frac{1}{B(\alpha, \beta)}
\]

Or

\[
\left( \frac{x + y}{x} \right) = \frac{1}{B(x + 1, y + 1)}
\]

Why all those silly +/− 1’s? Ask the Gamma.

**The random variables**

- \(X\) is a beta distribution
- \(Y|X\) is a binomial distribution with \(p = X\).
- Joint is

\[
f(x, i) = m(i|x)B(\alpha, \beta, x)
= \binom{n}{i} x^i(1 - x)^j \frac{1}{B(\alpha, \beta)} x^{\alpha - 1}(1 - x)^{\beta - 1}
= \binom{n}{i} \frac{1}{B(\alpha, \beta)} x^{\alpha + i - 1}(1 - x)^{\beta + j - 1}.
\]

- Distribution of \(Y\) is:

\[
m(i) = \int_0^1 m(i|x)B(\alpha, \beta, x) \, dx
= \binom{n}{i} \frac{1}{B(\alpha, \beta)} \int_0^1 x^{\alpha + i - 1}(1 - x)^{\beta + j - 1} \, dx
= \binom{n}{i} \frac{B(\alpha + i, \beta + j)}{B(\alpha, \beta)}.
\]

- So \(f(x|Y = i) = f(x, i)\) is just a beta! Everything cancels!