Administrivia

- homework questions?

Random variables

We have, the idea of $X$, and $Y$ that are random variables. We can do

\[
X + Y \\
X \ast Y \\
3 \ast X \\
X^2 \\
X^3 \\
e^X
\]

at least conceptually. But we don’t know what a Random variable is.

definition by usage

What is the minimal collection of operations that we need?
• scaling (changing units)
• typical value (spoiler alert: called expectation)
• sum
• product

But the list could go on with \( \sin(X) \), \( e^X \), \( \ln(X) \), etc. Yikes! Or does it have to?

**We could stop here and hope**

Called Banach Algebra. Game plan:

• quick review of linear algebra for finite dimensions (i.e. matrixes)
• longer introduction into infinite dimensional linear spaces
• work quickly through functional analysis
• Introduce banach algebras and von neuman algebras (might as well throw in QM while we are here)
• Now show that \( e^X = 1 + X + X^2/2 + X^3/3! + ... \) makes sense
• YIKES!!!

**A better way**

• Sample space: \( \Omega = \) list of possible outcomes
• Events \( A, B, \ldots \), are subsets of \( \Omega \)
• outcomes, \( \omega_1, \omega_2 \), are elements of \( \Omega \)
• Probability: $P$ takes subset of $\Omega$
• distribution function $m$ takes elements of $\Omega$
• Rules
  - $m(\omega) \geq 0$
  - $\sum m(\omega) = 1$
• Relationship to the $P$ operator
  \[ P(A) \equiv \sum_{\omega \in A} m(\omega) \]

Venn diagrams, etc
Draw union, intersection

But we have lost our random variables

definition: A random variable is a function from $\Omega$ to the real line.
  Easy example: if the $\omega$ are real numbers to start with, then the identity is a good function to work with

Theorem 1

1. $P(E) \geq 0$
2. $P(\Omega) = 1$
3. If $E \subset F \subset \Omega$ then $P(E) \leq P(F)$
4. Sums, for disjoint
5. Complement rule
Extensions: Finite sums

Key theorem:

**Theorem 2 (Summing over disjoint events)**  

*First version*

\[ P(E) = \sum P(E_i) \]

If \( E = \bigcup E_i \) and \( E_i \cap E_j = \emptyset \).

*Second version*

\[ P(E) = \sum P(E \cap A_i) \]

If \( \Omega = \bigcup A_i \) and \( A_i \cap A_j = \emptyset \).

**Simulations**

For finite sample spaces, you can just have the compute pick one out of the list. Can do it exactly. Describe arithmetic coding.

**Infinite sample spaces**

**Example 1** Toss a coin until the first time it comes up heads.

\[ \Omega = \{1, 2, 3, \ldots\} . \]

So, \( m(n) = 1/2^n \). Hopefully:

\[ \sum_{\omega} m(\omega) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1 . \]


\[ 1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r} , \]
If you memorize it, you have to remember whether it is \( r/(1 - r) \) or \( 1/(1 - r) \). So, let’s derive it instead:

\[
S \equiv 1 + r + r^2 + r^3 + \cdots
\]

\[
S = 1 + r(1 + r + r^2 + r^3 + \cdots)
\]

\[
S = 1 + rS
\]

\[
S(1 - r) = 1
\]

\[
S = 1/(1 - r)
\]

So works for \( 1/2 \).

Math geek question: Why isn’t my proof a happy proof? Works even if \( r = -1 \). So, “\( 1 - 1 + 1 - 1 + \cdots = 1/2 \)” Huh?