STAT 433: MIDTERM SOLUTIONS

QUESTION 1
(a)

$$\begin{aligned} E(Q_t) &= E\left(M_t^2 - t\sigma^2\right) \\ &= E\left(\left(\sum_{i=1}^{t}\eta_i\right)^2 - t\sigma^2\right) \\ &= E\left(\sum_{i=1}^{t}\eta_i^2 + \sum\sum_{i\neq j}\eta_i\eta_j\right) - t\sigma^2 \\ &= t\sigma^2 - t\sigma^2 \end{aligned}$$

(b)

$$\begin{aligned} E(Q_t|Q_{t-1}) &= E(M_t^2 - t\sigma^2|Q_{t-1}) \\ &= E\left([M_{t-1}+\eta_t]^2 - (t-1)\sigma^2 - \sigma^2|Q_{t-1}\right) \\ &= E\left(M_{t-1}^2 - (t-1)\sigma^2|Q_{t-1}\right) + E\left(\eta_t^2 - \sigma^2|Q_{t-1}\right) \\ &= M_{t-1}^2 - (t-1)\sigma^2 = Q_{t-1} \end{aligned}$$

(c)

$$\begin{aligned} E(Q_t|Q_{t-1}) &= E\left([M_{t-1}+\eta_t]^2 - t\sigma^2|Q_{t-1},Q_{t-2},\ldots,Q_0\right) \\ &= E\left(M_{t-1}^2 + 2M_{t-1}\eta_t + \eta_t^2 - (t-1)\sigma^2 - \sigma^2|Q_{t-1},Q_{t-2},\ldots,Q_0\right) \\ &= E\left(Q_{t-1} + 2M_{t-1}\eta_t + \eta_t^2 - \sigma^2|Q_{t-1},Q_{t-2},\ldots,Q_0\right) \\ &= Q_{t-1} + 2E\left(E\left(M_{t-1}\eta_t + \eta_t^2 - \sigma^2|M_{t-1},Q_{t-1},Q_{t-2},\ldots,Q_0\right)|Q_{t-1},Q_{t-2},\ldots,Q_0\right) \\ &= Q_{t-1} + 2E\left(M_{t-1}E\left(\eta_t + \eta_t^2 - \sigma^2|M_{t-1},Q_{t-1},Q_{t-2},\ldots,Q_0\right)|Q_{t-1},Q_{t-2},\ldots,Q_0\right) \\ &= Q_{t-1} + 2E\left(M_{t-1}\cdot 0|Q_{t-1},Q_{t-2},\ldots,Q_0\right) \\ &= Q_{t-1} \end{aligned}$$

(d) In order to demonstrate that this is Markovian, we need to make a statement about the joint probability distribution. I.e. show $P(Q_t|Q_{t-1} = P(Q_{t-j}|Q_{t-j-1})$
We have made no such statement and hence do not know whether or not this process is Markovian.
However, since $E(|Q_t|) < \infty$ and from (b) we know it is a martingale

QUESTION 2
(a)

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$$P = \begin{bmatrix} .1 & .5 & .1 & .3 \\ .2 & .4 & .1 & .3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)
If we let $\tau = \min\{n \geq 0; X_n = 3 \text{ or } X_n = 4\}$ and $u_i = P(X_T = 4|X_1 = i)$ then the first step equations are

$$\begin{aligned} u_1 &= 0.1u_1 + .5u_2 + .3 \\ u_2 &= 0.2u_1 + .4u_2 + .3 \\ u_3 &= 0 \\ u_4 &= 1 \end{aligned}$$

QUESTION 3
(a)

Step 1
$E(Z_n) = E\left(X_n \cdot 2^{-n}\right) = 2^{-n}E(X_n) = 2^{-n}\mu^n = 2^{-n}2^n = 1 < \infty$
Step 2
$E(Z_n|Z_{n-1}, \ldots, Z_0) = 2^{-n}E(X_n|X_{n-1}, \ldots, X_0) = 2^{n-1}2X_{n-1} = Z_{n-1}$

(b) From the maximal inequality we have

$$\begin{aligned} P\left(\max_{0 \leq i \leq n} X_i \geq 100 \cdot 2^n\right) &= P\left(\max_{0 \leq i \leq n} Z_i \geq 100\right) \\ &\leq \frac{E(Z_0)}{100} = \frac{1}{100} \end{aligned}$$

QUESTION 4

(a)
The states are as follows

State	Number of		
	Red balls	Green balls	Blue balls
0	2	1	0
1	2	0	1
2	1	1	1
3	1	0	2
4	0	1	2
5	0	0	3
6	1 blue	ball is	drawn

(b) The transition matrix would be

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) We seek a solution to the following system of equations

$$\begin{aligned} v_0 &= 1/3v_1 + 2/3v_2 + 1 \\ v_1 &= 2/3v_3 + 1 \\ v_2 &= 1/3v_3 + 1/3v_4 + 1 \\ v_3 &= 1/3v_5 + 1 \\ v_4 &= 1/3v_5 + 1 \\ v_5 &= 1 \\ v_6 &= 0 \end{aligned}$$

which leads to $v_0 = \frac{26}{9}$

QUESTION 5

(a) State 3 is the absorbing state

(b)

$$\begin{aligned} u_i &= P(X_T = 3 | X_0 = i) \\ u_1 &= au_1 + bu_2 \\ u_2 &= cU_2 + d \\ u_3 &= 1 \\ &\Downarrow \\ u_2 - cu_2 &= d \Rightarrow u_2 = \frac{d}{1-c} = P(X_T = 3 | X_0 = 2) \end{aligned}$$

(c)

$$\begin{aligned} u_1 &= au_1 + bu_2 \\ (1-a)u_1 &= \frac{bd}{1-c} \\ u_1 &= \frac{bd}{(1-a)(1-c)} = \underbrace{\frac{bd}{(1-a)(1-c)}}_{b=1-a, d=1-c} = \frac{bd}{bd} = 1 \end{aligned}$$

(d) **(e)** Remember from part (a) that state 3 is the absorbing state so eventually ......

$$P^T \approx \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and an appropriate bound would be

$$P^T \approx \begin{bmatrix} a^T & (1-bd)^{T/2} & 1 \\ 0 & c^T & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

QUESTION 6

If a team of gamblers come into a casino everyday to gamble on the expression "TO BE OR NOT TO BE", with the casino laying fair odds 27 to 1, the expected winnings of the casino is zero.

$$\begin{aligned} 0 &= E[\tau] - \underbrace{27^{18}}_{\text{"TO BE OR NOT TO BE" winner}} - \underbrace{27^{5}}_{\text{"TO BE " winner}} \\ E[\tau] &= 27^{18} + 27^{5} \end{aligned}$$