Be sure to show your work. It is great if you can guess the right answer. But to call it mathematics, you need to be able to show why your guess is correct. Hence provide justifications!

1. (20 pts) We have discussed many “Markov-martingale”: e.g. random walks, exponential random walks, branching processes with $\mu = 1$. One that we haven’t discussed is called the “square martingale.” Let’s discuss it now.

Suppose that $M_t = \sum_{i=1}^{t} \eta_i$ where the $\eta_i$’s are all independent and identically distributed with mean 0 and variance $\sigma^2$. Take $M_0 = 0$. Let $Q_t = M_t^2 - t\sigma^2$. Obviously $Q_0 = 0$.

(a) Derive $E(Q_t)$? (i.e. Derive means “show your reasoning.”)
(b) Derive $E(Q_t|Q_{t-1})$?
(c) Derive $E(Q_t|Q_{t-1}, Q_{t-2}, \ldots, Q_0)$?
(d) Comment on whether the above 3 parts show that $M_t$ is Markovian? Do they show that $M_t$ is a martingale?

2. (20 pts) Consider the following transition matrix between states 1, 2, 3 and 4:

\[
P = \begin{bmatrix}
.1 & .5 & .1 & .3 \\
.2 & .4 & .1 & .3 \\
0 & .4 & .4 & .3 \\
0 & .2 & .4 & .1
\end{bmatrix}
\]

Suppose the process will actually be stopped when it arrives in either of the last two states.

(a) Write down the transition matrix for this modified processes.
(b) If it reaches state 3 first, we “lose”. If it reaches state 4 first, we “win.” Write down the first step equations for the probability of winning given we start in state $i$.

3. (20 pts) Consider a flu epidemic that “doubles” every week. In other words, we will model the number of people with the flu as a branching process (called $X_t$) with mean family size $\mu = 2$. As usual, take $X_0 = 1$.

(a) Show that $Z_n = X_n/2^n$ is a non-negative martingale.
(b) Suppose the CDC (Center for Disease Control) predicts that in week $n$ there will be fewer than $100 \times 2^n$ flu cases in the US. What is the probability that at some point during the flu season, this prediction will be wrong?
4. (20 pts) Suppose an urn contains two red and one green ball. Each time a ball is removed, it is replaced with a blue ball. The process is stopped the first time a blue ball is drawn.

(a) (5 pts) List the possible states the system could be in.

(b) (10 pts) Write down the transition matrix.

(c) (5 pts) How many rounds do we expect this process to run for?

5. (20 pts) Consider the following transition matrix:

\[ P = \begin{bmatrix} a & b & 0 \\ 0 & c & d \\ 0 & 0 & 1 \end{bmatrix} \]

where \( a > 0, b > 0, c > 0, \) and \( d > 0. \)

(a) If the states are called state 1, state 2, and state 3, which of these three states are absorbing?

(b) Write down the first step equation that solves the probability of state 2 eventually going to state 3. What is the solution?

(c) What is the probability that starting from state 1, the process will eventually be in state 3?

(d) Approximately, what will \( P^T \) look like for large \( T \)?

(e) (bonus = 5 pt) Give a non-trivial upper bound on \( P^T \).

6. (For fun only. Or 5 pts) Modern computers are supposed to be very, very fast. So they should be able to simulate the problem of “a million monkeys will eventually type out all the works of Shakespeare.” As a simple starting point, lets simulate random draws from the 27 symbols, “A”, “B”, “C”, . . . , “Z”, “ “.

(a) How long will we have to wait for the 18 character phrase “TO BE OR NOT TO BE”?

(b) What comes next? (In the Shakespeare, not the monkeys!)