Dean Foster **Stat 433: Midterm Exam** February 22, 2008

Instructions:

- Be sure to show your work. It is great if you can guess the right answer. But to call it mathematics, you need to be able to show why your guess is correct. Hence provide justifications!
- No calculators.
- You may use a single page of notes.

1. (20 pts) Joe Math walks into a fair gambling institution. He starts out with 10 dollars in his pocket. Each round he bets on a 50/50 fair gamble. But since he believes in the theory of "runs" he increases the amount he bets by one dollar whenever he wins. After he loses, he goes back to betting one dollar. So for example, if he were to win, win, lose, win, his bet amounts would be,

   (i) bet $1, which he wins

   (ii) bet $2, which he wins,

   (iii) bet $3, which he loses,

   (iv) bet $1, which he wins,

   (v) bet $2, ...

   Obviously his total wealth W_t would be,

   (i) W_0 =$10, (bets $1, and wins)

   (ii) W_1 =$11 (bets $2, and wins)

   (iii) W_2 =$13 (bets $3, and loses)

   (iv) W_3 =$10 (bets $1, and wins)

   (v) W_4 =$11 (bets $2, ...)

   If he runs out of money, he stops betting. If he ends up making more than $100, he stop betting.

   (a) Prove W_t is a non-negative martingale.

   (b) Using the martingale maximal inequality bound the probability that he ever has $100 or more.

   (c) Is W_t a Markov chain? If so, how many states does it have, if not, say why not.

2. (20 pts) In a very simple children's game, we can model a part of the game as being a four state Markov chain:

$$P = \begin{bmatrix} .2 & .2 & .2 & .2 & .2 \\ 0 & .25 & .25 & .25 & .25 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where the four states are called A, B, C, Win, and Lose. In states A, B, and C, the player wins a dollar for every round. If they end up in "Win" they get to keep all the money they have collected. If they end up in state "Lose" they keep nothing. We are interested how much money they will win on average in this game.

(a) Write down the first step equations for the probability of ending up in "Win." Solve these equations.

(b) Write down the first step equations for how much money they win at the end of the day. Solve these equations.

3. (20 pts) Consider a 5 state random walk with absorbing boundaries.

(a) Write down the transition matrix.

(b) Suppose that you are paid a dollar ever time you land on state 3 (the middle state). What is the expected amount you win if you started in each of the 5 possible states?

4. (20 pts) Consider a sequence of coin tosses.

(a) On average how long will it take to get the pattern H,H,T,H?

(b) On average how long will it take to get the pattern T,H,H,H?

5. (20 pts) Consider a very simple branching process. Each organism has a $1 - \epsilon$ change of having a single child, and a ϵ chance of having no children. (Of course, it dies after reproducing.) Assume the process starts with 6 organisms.

(a) Write down the transition matrix for the process.

(b) What is the probability of the process going extinct in 10 rounds if it starts with a single organism?

(c) Using the previous part, compute the probability of going extinct in 10 rounds if the process starts with 6 organisms.