The correct finish to the proof

Suppose $X_1, X_2, \ldots, X_n$ form a martingale, and $X_i \geq 0$ for all $i$, then

$$P(\max_{i=1,n} X_i \geq \lambda) \leq E(X_1)/\lambda.$$ 

Proof: Here is what we did in class:

$$P(\max_{i=1,n} X_i \geq \lambda) = \sum_{i=1}^{n} P((X_i \geq \lambda) \cap (X_j < \lambda \text{ for } j < i))$$

$$= \sum_{i=1}^{n} E(I_{(X_i \geq \lambda) \cap (X_j < \lambda \text{ for } j < i)})$$

$$\leq \sum_{i=1}^{n} E((X_i/\lambda)I_{(X_i \geq \lambda) \cap (X_j < \lambda \text{ for } j < i)})$$

Now we need to condition the other direction than I was conditioning in class. So we need to use that $E(X_m|X_1, X_2, \ldots, X_i) = X_i$.

$$P(\max_{i=1,n} X_i \geq \lambda) \leq \ldots$$

$$= \sum_{i=1}^{n} E \left( E \left( X_m|X_1, X_2, \ldots, X_i \right) / \lambda I_{(X_i \geq \lambda) \cap (X_j < \lambda \text{ for } j < i)} \right)$$

$$= \sum_{i=1}^{n} E \left( E \left( X_m/\lambda I_{(X_i \geq \lambda) \cap (X_j < \lambda \text{ for } j < i)} \right) \right)$$

$$= E \left( E \left( (X_m/\lambda) \sum_{i=1}^{n} I_{(X_i \geq \lambda) \cap (X_j < \lambda \text{ for } j < i)} \right) \right)$$

$$\leq E(E(X_m/\lambda)) = E(X_m/\lambda) = E(X_1)/\lambda$$

NOTE: This is basically the proof that is given in the book, but working backwards.