Course No. Stat 433
April 7, 2008

Homework 8 Solution

Comments from the grader:

- These are only partial solutions. We selected questions which were problematic to most of the class or are of particular interest.
- The maximum grade for this homework assignment is 10.
- Your solution should contain explanations and not only final answers. Points will be deducted if partial solutions are submitted.
- Please save a copy of your work and submit the original. Write your name and email on top of the first page.
- if you notice a typo in the solution file or have a problem with the homework grading please email: sivana@wharton.upenn.edu

Page 231 Question 2.2

a.Let X count the number of operating components and Y an indicator which takes on a value one when a component is repaired for 1 day. The markov chain states are as follows:

1. State 1: $X = 2, Y = 0$
2. State 2: $X = 1, Y = 0$
3. State 3: $X = 1, Y = 1$
4. State 4: $X = 0, Y = 1$
5. State 5: $X = 0, Y = 0$

The appropriate transition matrix is:

$$P = \begin{pmatrix} (1-\alpha)^2 & 2\alpha(1-\alpha) & 0 & 0 & \alpha^2 \\ 0 & 0 & 1-\beta & \beta & 0 \\ 1-\beta & \beta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

b. Solve the following equations:

$$\begin{aligned} (1-\alpha)^2\pi_0 + (1-\beta)\pi_2 &= \pi_0 \\ 2\alpha(1-\alpha)\pi_0 + \beta\pi_2 + \pi_3 &= \pi_1 \\ (1-\beta)\pi_1 &= \pi_2 \\ \beta\pi_1 + \pi_4 &= \pi_3 \\ \alpha^2\pi_0 &= \pi_4 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}$$

The fraction of time the system is operating is $\pi_0 + \pi_1 + \pi_2 \approx 0.95$.

Page 233 Question 2.6
We begin by proving the X_n defines a Markov chain.
Consider the following 4 situations:

- $P(X_{n+1} = 0|X_0, \ldots, X_n = 1) = P(X_{n+1} = 0|X_n = 1) = p$ from the problem definition ("on a given day the computer fails with probability p")
- $P(X_{n+1} = 1|X_0, \ldots, X_n = 1) = P(X_{n+1} = 1|X_n = 1) = q$ using the same logic as before.
- $P(X_{n+1} = 0|X_0, \ldots, X_n = 0) = P(N > k + 1|N > k)$ for some time k . Since N has a geometric distribution we know that $P(N > k + 1|N > k) = P(N > 1) = (1 - \beta) = \alpha$. Since this is independent of k we arrive to the conclusion that $P(X_{n+1} = 0|X_0, \ldots, X_n = 0) = P(X_{n+1} = 0|X_n = 0) = \alpha$.
- $P(X_{n+1} = 1|X_0, \ldots, X_n = 0) = P(X_{n+1} = 1|X_n = 0) = \beta$ using the same logic as before.

Therefore, X_n is a Markov Chain with P, the transition matrix, which is indicated in the question. Since $\beta \in (0, 1)$ P must be regular and thus by solving $\pi P = \pi$ we can obtain the limit distribution $\pi = [\frac{p}{p+\beta}, \frac{\beta}{p+\beta}]'$.

Page 243 Question 3.3
a. The classes are:

1. $\{0, 2\}$ since $p_{0,2} > 0$ and $p_{2,0} > 0$.
2. $\{4, 5\}$ since $p_{4,5} > 0$ and $p_{5,4} > 0$.
3. $\{1, 3\}$ since $p_{1,3} > 0$ and $p_{3,1} > 0$.

b.The classes are:

1. $\{1, 2\}$ since $p_{1,2} > 0$ and $p_{2,1} > 0$.
2. $\{4, 3\}$ since $p_{4,3} > 0$ and $p_{3,4} > 0$.
3. $\{0\}$ and $\{5\}$ since they are both absorbing states.

Page 245 Question 3.1

a.

$$\begin{aligned} f_{00}^{(0)} &= 0 \\ f_{00}^{(1)} &= 1-a \\ f_{00}^{(n)} &= a(1-b)^{n-2}b \text{ for } n>1 \end{aligned}$$

b.

$$\begin{aligned} \sum_{k=1}^{n} f_{00}^{(k)} p_{00}^{(n-k)} &= (1-a)\frac{b+a(1-a-b)^{n-1}}{a+b} + \sum_{k=2}^{n} a(1-b)^{k-2}b \cdot \frac{b+a(1-a-b)^{n-k}}{a+b} \\ &= \frac{b}{a+b} + \frac{a(1-a-b)^{n}}{a+b} \end{aligned}$$

The last transition requires tedious algebra operations (using definition of geometric sums) but you were expected to perform them and get the last result.