Course No. Stat 433
April 7, 2008

Homework 7 Solution

Comments from the grader:

- These are only partial solutions. We selected questions which were problematic to most of the class.
- The maximum grade for this homework assignment is 10.
- Your solution should contain explanations and not only final answers. Points will be deducted if partial solutions are submitted.
- Please save a copy of your work and submit the original. Write your name and email on top of the first page.
- if you notice a typo in the solution file or have a problem with the homework grading please email: sivana@wharton.upenn.edu

Page 211 Question 1
Let X indicate the number of balls in A.
The appropriate transition matrix is:

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

First check and see that P is regular (i.e., P^8 is a matrix where all the entries are positive). Since the rows and columns of P both add up to 1 it means that P is doubly stochastic. Hence, the limit distribution is $\pi = (\frac{1}{6}, \ldots, \frac{1}{6})'$. This means that the markov chain spends $\frac{1}{6}$ of its time in state 0 (or any other state).

Page 211 Question 6

1. $lim_{n \longrightarrow \infty} P(X_{n+1} = j | X_0 = i) = \pi_j$
2. $lim_{n \longrightarrow \infty} P(X_n = k, X_{n+1} = j | X_0 = i) = \pi_k \cdot p_{kj}$
3. $lim_{n \longrightarrow \infty} P(X_{n-1} = k, X_n = j | X_0 = i) = \pi_k \cdot p_{kj}$

Page 211 Question 10
P is regular and has $N+1$ states (from 0 to N). P is also doubly stochastic and hence its limit distribution is $\pi = (\frac{1}{N+1}, \ldots, \frac{1}{N+1})'$

Page 211 Question 12
a.Since Π is the stationary distribution matrix and $P = Q + \Pi$ we know that:

- $\Pi P = \Pi$
- $\Pi^2 = \Pi$
- $\Pi Q = \Pi P - \Pi^2 = 0$

Hence since $\Pi Q = 0$ and $\Pi^n = Pi$

$$\begin{aligned} P^n &= (Q + \Pi)^n \\ &= Q^n + \Pi^n \\ &= Q^n + \Pi \end{aligned}$$

b. First check and see that P is regular (since P^2 has positive entries). Hence by solving $\pi P = \pi$ we get the limit distribution which is $\pi = (0.25, 0.5, 0.25)$. Which means that

$$\Pi = \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.5 & 0.25 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0.25 & 0 & -0.25 \\ 0 & 0 & 0 \\ -0.25 & 0 & 0.25 \end{pmatrix}$$

Hence,

$$Q^n = \frac{1}{2^{n+1}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$P^n = \begin{pmatrix} 0.25 + \frac{1}{2^{n+1}} & 0.5 & 0.25 - \frac{1}{2^{n+1}} \\ 0.25 & 0.5 & 0.25 \\ -\frac{1}{2^{n+1}} + 0.25 & 0.5 & \frac{1}{2^{n+1}} + 0.25 \end{pmatrix}$$