Course No. Stat 433
February 24, 2008

# Homework 4 Solution

Comments from the grader:

- These are only partial solutions. We selected questions which were problematic to most of the class.
- The maximum grade for this homework assignment is 10.
- Your solution should contain explanations and not only final answers. Points will be deducted if partial solutions are submitted.
- Please save a copy of your work and submit the original. Write your name and email on top of the first page.
- if you notice a typo in the solution file or have a problem with the homework grading please email: sivana@wharton.upenn.edu

**Page 130 Question 4.1**
Our Markov Chain consists of 4 states:

1. State 0: All the flips up till now cannot possibly lead to the pattern HHT. If the next coin flip is a H we progress to state 1 else we stay in this state.
2. State 1: The current flip is H. If the next flip is H we continue to state 2. If the next flip is T we go back to state 0.
3. State 2: The current flip is H and the flip before was H. If the next flip is T we continue to state 2. If the next flip is H we stay in this state.
4. State 3: The current flip is T and the last two flips were H. Once it reaches state 3 the Markov Chain remains in this state.

The appropriate transition matrix is:

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 1 \end{pmatrix}$$

Let $T = min\{n \geq 0; X_n = 3\}$ and $u_i = E(T|X_0 = i)$. The first step analysis yields the following equations:

$$\begin{aligned} v_0 &= 1 + 0.5v_0 + 0.5v_1 \\ v_1 &= 1 + 0.5v_0 + 0.5v_2 \\ v_2 &= 1 + 0.5v_2 + 0.5v_3 \end{aligned}$$

If we start at state 3 then the mean time until we first get these is 0, hence $v_3 = 0$. Solving the above system of equations reveals that $v_0 = 8$.
A similar Markov Chain can be defined for the second part of the question in the following manner.

1. State 0: All the flips up till now cannot possibly lead to the pattern HTH. If the next coin flip is a H we progress to state 1 else we stay in this state.

2. State 1: The current flip is H. If the next flip is T we continue to state 2. If the next flip is H we stay in state 1.

3. State 2: The current flip is T and the flip before was H. If the next flip is H we continue to state 2. If the next flip is T we go back to state 0.

4. State 3: The current flip is H and the last two flips were HT. Once it reaches state 3 the Markov Chain remains in this state.

The appropriate transition matrix is:

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 1 \end{pmatrix}$$

Let $T = min\{n \geq 0; X_n = 3\}$ and $u_i = E(T|X_0 = i)$. The first step analysis yields the following equations:

$$\begin{aligned} v_0 &= 1 + 0.5v_0 + 0.5v_1 \\ v_1 &= 1 + 0.5v_1 + 0.5v_2 \\ v_2 &= 1 + 0.5v_0 + 0.5v_3 \end{aligned}$$

If we start at state 3 then the mean time until we first get these is 0, hence $v_3 = 0$. Solving the above system of equations reveals that $v_0 = 10$.
From this analysis one can see that the average time till we get a pattern of HHT is shorter than the average time till we reach HTH. This makes sense because once the first Markov Chain reaches state 2 it either stays there or advances while the second Markov Chain can return to state 0 even when it reaches state 2.

**Page 132 Question 4.7**

$$\begin{aligned} h_i &= \sum_{n=0}^{\infty} E[\beta^n c(X_n)|X_0 = i] \\ &= c(i) + \sum_{n=1}^{\infty} E[\beta^n c(X_n)|X_0 = i] \end{aligned}$$

$$
\begin{aligned}
&= c(i) + \sum_{n=1}^{\infty} \sum_{j} E[\beta^n c(X_n)|X_0 = i, X_1 = j] P(X_1 = j|X_0 = i) \\
&= c(i) + \sum_{n=1}^{\infty} \sum_{j} E[\beta^n c(X_n)|X_1 = j] P_{ij} \\
&= c(i) + \sum_{j} \beta \sum_{n=1}^{\infty} E[\beta^{n-1} c(X_n)|X_1 = j] p_{ij} \\
&= c(i) + \sum_{j} \beta \sum_{m=0}^{\infty} E[\beta^{m} c(X_m)|X_0 = j] p_{ij} \\
&= c(i) + \sum_{j} \beta \sum_{m=0}^{\infty} h_j p_{ij}
\end{aligned}
$$

where $m = n - 1$ and we just use $m$ as a dummy index.

**Page 134 Question 4.17**

Define $u_i = E(s^T|X_0 = i)$ then we need to find $u_0$. Using first step analysis we have the following equations:

$$
\begin{aligned}
u_0 &= 0.7su_0 + 0.3su_1 \\
u_1 &= 0.6su_1 + 0.4su_2
\end{aligned}
$$

Notice that $u_2 = 1$ since $E(s^T|X_0 = 2) = s^0 = 1$.
By solving the above set of equations we can conclude that

$$
u_0 = \frac{0.12s^2}{(1 - 0.7s)(1 - 0.6s)}
$$

Some students wrote that $E(g(X)) = g(E(X))$ but this is not generally true unless the function $g(X)$ is a linear function. In our case $g(X) = s^X$ which is not a linear function.