Course No. Stat 433
February 24, 2008

Solution Homework 2

Comments from the grader:

- These are only partial solutions. We selected questions which were problematic to most of the class.
- The maximum grade for this homework assignment is 10.
- Your solution should contain explanations and not only final answers. Points will be deducted if partial solutions are submitted.
- Please save a copy of your work and submit the original. Write your name and email on top of the first page.
- if you notice a typo or have a problem with the homework grading please email: sivana@wharton.upenn.edu

Comments regarding this assignment: The definition of a martingale consists of two parts (see page 87 in the text book). Most of you did not prove the first part, i.e. $E|X_n| < \infty$. If you did not prove this throughout the assignment you lost 1 point.

Page 94 Question 1

- Let $X = X_{n+2}$, $Y = X_{n+1}$ and $Z = X_0, \ldots, X_n$. From these definitions the first identity is proven.
- Hence if X_n is a martingale then:

$$\begin{aligned} E[X_{n+2}|X_0, \ldots, X_n] & \\ &= E[E[X_{n+2}|X_0, \ldots, X_{n+1}]|X_0, \ldots, X_n] \\ &= E[X_{n+1}|X_0, \ldots, X_n] = X_n \end{aligned}$$

Page 94 Question 2 We need to verify that

1. $E|X_n| < \infty$
2. $E[X_{n+1}|X_0, \ldots, X_n] = X_n$.

For the first part notice that X_n has non-negative values. Hence, $E|X_n| = E(X_n)$.

$$\begin{aligned} E|X_n| &= E[X_n] \\ &= E[2^n U_1 \dots U_n] \\ &= 2^n \prod_{i=1}^{n} E[U_i] \\ &= 1 < \infty \end{aligned}$$

The second part prove is as follows:

$$\begin{aligned} E[X_{n+1}|X_0, \dots, X_n] &= E[2^{n+1} U_1 \dots U_{n+1} | 1, 2U_1, \dots, 2^n U_1 \dots U_n] \\ &= E[2^{n+1} U_1 \dots U_{n+1} | 2^n U_1 \dots U_n] \\ &= 2^n U_1 \dots U_n \cdot E[2U_{n+1} | 2^n U_1 \dots U_n] \\ &= 2^n U_1 \dots U_n \cdot E[2U_{n+1}] = 2^n U_1 \dots U_n = X_n \end{aligned}$$

Page 94 Question 4 Again we need to prove the two parts of the martingale definition:

- First $E[|X_n|] = E[X_n]$ since X_n can only have non-negative values. Hence,

$$\begin{aligned} E[X_n] &= E[p^{-n} \cdot \xi_1 \cdots \xi_n] \\ &= p^{-n} \cdot \prod_{i=1}^{n} E[\xi_i] \\ &= p^{-n} \cdot p^n \\ &= 1 < \infty \end{aligned}$$

- The following is the proof for the second part of the martingale definition:

$$\begin{aligned} & E[X_{n+1}|X_0, \dots, X_n] \\ &\quad = E[p^{-(n+1)} \cdot \xi_1 \cdots \xi_{n+1} | X_0, \dots, X_n] \\ &\quad = E[p^{-1} \cdot X_n \cdot \xi_{n+1} | X_0, \dots, X_n] \\ &\quad = p^{-1} \cdot X_n \cdot p \\ &\quad = X_n \end{aligned}$$

A lot of the students ignored the second part of this question about the convergence of X_n (and lost points as a result). Notice that for any $\epsilon > 0$ the following holds:

$$\begin{aligned} P(|X_n - 0| > \epsilon) &= P(X_n > \epsilon) \\ &= P(\xi_1 \cdots \xi_n \neq 0) \\ &= P(\xi_1 \cdots \xi_n = 1) \\ &= p^{-n} \end{aligned}$$

By letting n go to infinity we know that $P(|X_n - 0| > \epsilon) \to 0$ for any $\epsilon > 0$. This means that X_n converges in probability to 0.

Page 99 Question 1 The matrix has the following structure:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Consider the second row probabilities. Given that the current period has a single diseased person, the probability that there are 2 diseased people in the next period is $P(X_{n+1} = 2|X_n = 1) = \alpha \cdot P(\text{the diseased person interacts})$. Which means $P(X_{n+1} = 2|X_n = 1) = 0.1 \cdot (1 - 0.6) = 0.04$. Hence the complimentary probability is 0.96. Following the same logic yields the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.96 & 0.04 & 0 & 0 & 0 \\ 0 & 0 & 0.96 & 0.04 & 0 & 0 \\ 0 & 0 & 0 & 0.96 & 0.04 & 0 \\ 0 & 0 & 0 & 0 & 0.96 & 0.04 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$