Course No. Stat 433
May 5, 2008

# Homework 11 Solution

Comments from the grader:

- These are only partial solutions. We selected questions which were problematic to most of the class or are of particular interest.
- The maximum grade for this homework assignment is 10.
- Your solution should contain explanations and not only final answers. Points will be deducted if partial solutions are submitted.
- Please save a copy of your work and submit the original. Write your name and email on top of the first page.
- if you notice a typo in the solution file or have a problem with the homework grading please email: sivana@wharton.upenn.edu

**Question 2.4**

a.

$$\begin{aligned}
\mu_k &= \lim_{h\to\infty} \frac{P(X(t+h)=k-1|X(t)=k)}{h} \\
&= \lim_{h\to\infty} \frac{\binom{k}{1}\left[\binom{M-(N-k)}{1}(\theta+o(h))\right]\cdot\left[1-\left(\binom{M-(N-k)}{1}(\theta+o(h))\right)\right]^{k-1}}{h}
\end{aligned}$$

Now you need to work out the algebra and some limits (which all go to zero) and you are left with the result $\mu_k = \theta k(M-(N-k))$. Some students reached the same result using their intuition and got full marks.

b.

$$\begin{aligned}
E(T) &= E(\sum_{i=1}^{N} S_i) \\
&= \sum_{i=1}^{N} E(S_i) \\
&= \sum_{i=1}^{N} \frac{1}{\theta i(M-(N-i))} \\
&= \frac{1}{\theta}\frac{1}{(M-N)}\sum_{i=1}^{N}(\frac{1}{i}-\frac{1}{M-N+i})) \\
&= \frac{1}{\theta}\frac{1}{(M-N)}(\sum_{i=1}^{N}\frac{1}{i}-\sum_{j=M-N+1}^{M}\frac{1}{j})
\end{aligned}$$

**Question 3.2**
We can model this process by $\lambda_k = \frac{1}{2m_k}$ and $\mu_k = \frac{1}{2m_k}$. The boundaries assumptions are

$$\lambda_0 = \frac{1}{m_0} \quad \mu_0 = 0$$
$$\lambda_N = 0 \quad \mu_N = \frac{1}{m_N}$$

Please notice that $\mu$ and $\lambda$ are rates. Some students wrote $\lambda_0 = \mu_N = 1$ probably thinking of probabilities (like in discrete Markov Chain) however this may not be the case ($m_0$ might equal some other number, for example).

**Question 1.2**
$B(t)$ follows the normal distribution with mean zero and variance $t$. Hence,

$$\begin{aligned} E(e^{\lambda B(t)}) &= \int_{-\infty}^{\infty} e^{\lambda z} \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dz \\ &= e^{\frac{\lambda^2 t}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(z-\lambda t)^2}{2t}} dz \\ &= e^{\frac{\lambda^2 t}{2}} \end{aligned}$$

The last step is true since the expression written inside the integral is the normal density of a variable with mean $\lambda t$ and variance $t$. As a result the integral is just one. Some students use the result of the moment generating function of the normal distribution which was also a good way of solving this problem. Another good way was to use the mean value of the log-normal distribution.

**Question 1.5**
a.$M_\tau = 0$ implies that the random walk maximum value is zero. This is the same as saying that the random walk reaches $-a$ before it reaches 1 (so it can hit zero multiple times but never go above it). Using the formula on page 480 we know that $P(S_n \text{ reaches } -a \text{ before } 1 | S_0 = 0) = \frac{1}{1+a} = P(M_\tau = 0)$.
b.

$$\begin{aligned} P(M_\tau \geq 1) &= 1 - P(M_\tau = 0) \\ &= \frac{a}{1+a} \end{aligned}$$

Now lets examine the $P(M_\tau \geq 2)$.We know that the following equation is true:

$$P(M_\tau \geq 2) \quad = \quad P(M_\tau \geq 2|M_\tau \geq 1) \cdot P(M_\tau \geq 1)$$

Saying that we know that $M_\tau \geq 1$, is like shifting the starting point to 1 (instead of zero) and starting the process from there. Or in other words, $P(M_\tau \geq 2|M_\tau \geq 1)$ has the same probability as $P(M_\tau \geq 1|S_0 = 0)$.Hence,

$$P(M_\tau \geq 2) \quad = \quad \frac{a}{1+a} \cdot \frac{a}{1+a}$$

Using the same logic it is clear why $P(M_\tau \geq k) = (\frac{a}{1+a})^k$.
This implies that $M_\tau$ follows the geometric distribution.
c.

$$\begin{aligned}
P(M_\tau \geq k) &= (\frac{a\sqrt{n}}{1+a\sqrt{n}})^{k\sqrt{n}} \\
&= ((1-\frac{1}{1+a\sqrt{n}})^{a\sqrt{n}})^{\frac{k}{a}} \\
&\to \lim_{n\to\infty}((1-\frac{1}{1+a\sqrt{n}})^{a\sqrt{n}})^{\frac{k}{a}} \\
&= (e^{-1})^{\frac{k}{a}} \\
&= e^{-\frac{k}{a}}
\end{aligned}$$

Since $P(M_\tau \geq k) = e^{-\frac{k}{a}}$ this implies that $M_\tau$ follows the exponential distribution with parameter $\frac{1}{a}$ (if you don't see this just take the first derivative of this function to get the well-known density of the exponential distribution).