Course No. Stat 433
April 26, 2008

Homework 10 Solution

Comments from the grader:

- These are only partial solutions. We selected questions which were problematic to most of the class or are of particular interest.
- The maximum grade for this homework assignment is 50.
- Your solution should contain explanations and not only final answers. Points will be deducted if partial solutions are submitted.
- Please save a copy of your work and submit the original. Write your name and email on top of the first page.
- if you notice a typo in the solution file or have a problem with the homework grading please email: sivana@wharton.upenn.edu

Question 2.4
Each point falls in the interval [0,1) with probability $1/N$ and we have N points. The number of points in the interval is distributed binomially with parameters ($1/N$,N). Using the law of rare events we can conclude that as $N \to \infty$ the number of points in the interval S_N follows a Poisson distribution with $\lambda = \frac{1}{N} \cdot N = 1$.

Question 2.10
We know that

$$P(E(p) = X(p)) \geq 1 - p^2 \quad \Rightarrow \quad P(E(p) \neq X(p)) \leq p^2$$

Using equation 2.8 on page 285 in the book we know that for all $k \in I$ the following is true

$$|P(S_n = k) - P(X(\mu) = k)| \leq \sum_{k=1}^{n} P(E(p_k) \neq X(p_k)) \leq \sum_{k=1}^{n} p_k^2$$

Since this holds for all $k \in I$ the desired result holds.

Question 2.11
First notice that $\{X \in B\} = \{X \in B \ \cap Y \in B\} \cup \{X \in B \ \cap Y \notin B\}$. Using the same logic we know that
$\{Y \in B\} = \{Y \in B \ \cap X \in B\} \cup \{Y \in B \ \cap X \notin B\}$. Hence,

$$\begin{aligned}
&|P(X \in B) - P(Y \in B)| \\
&\qquad = |P(X \in B) + P(Y \in B) - P(X \in B) - P(Y \notin B)| \\
&\qquad = |P(X \in B \ \cap Y \in B) + P(X \in B \ \cap Y \notin B) \\
&\qquad - P(Y \in B \ \cap X \in B) - P(Y \in B \ \cap X \notin B) \\
&\qquad = |P(X \in B \ \cap Y \notin B) - P(Y \in B \ \cap X \notin B)
\end{aligned}$$

Since $\{X \in B \ \cap Y \notin B\} \subseteq \{X \neq Y\}$ the desired result follows.

Question 3.3
The two dimensional transformation of variables formula is

$$f_{S_0,S_1}(s_0, s_1) = f_{W_0,W_1}(w_0, w_1)det(J).$$

where J is the Jacobain matrix of (S_0, S_1) as a function of (W_0, W_1). In this example the determinant of the Jacobain is 1 and as a result the density is $\lambda^2 e^{-\lambda(s_0+s_1)} = \lambda e^{-\lambda s_0} \cdot \lambda e^{-\lambda s_1}$. Hence we see that it is just the joint distribution of two independent exponential random variables.

Question 3.8

$$\begin{aligned}
P(W_r = x|X(t) = n) &= \frac{\frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!} \cdot \frac{(\lambda(t-x))^{n-r} e^{-\lambda(t-x)}}{(n-r)!}}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}} \\
&= \begin{pmatrix} n \\ r \end{pmatrix} \cdot \frac{r}{t}(\frac{x}{t})^{r-1}(1 - \frac{x}{t})^{n-r}
\end{aligned}$$

A different that arrives to the same solution is to start from the conditional cumulative distribution function and take the derivative with respect to x.

Question 4.8

Using the iterated expectation rule we know that $E(Z(t)) = E(E(Z(t)|N(t)))$. First we will find $E(Z(t)|N(t))$.

$$\begin{aligned} E(Z(t)|N(t)) &= E(\sum_{k=1}^{N(t)} \theta_k(t)) \\ &= \sum_{k=1}^{N(t)} E(\theta_k(t)) \\ &= \sum_{k=1}^{N(t)} E(\xi_k e^{-\alpha(t-w_k)}) \\ &= E(\xi_1) \sum_{k=1}^{N(t)} E(e^{-\alpha(t-w_k)}) \\ &= E(\xi_1) \frac{N(t)}{\alpha t} \cdot (1 - e^{-\alpha t}) \end{aligned}$$

Hence, $E(Z(t)) = E(E(\xi_1) \frac{N(t)}{\alpha t} \cdot (1 - e^{-\alpha t})) = \frac{\lambda t E(\xi_1)}{\alpha t}(1 - e^{-\alpha t})$