STAT 433: FINAL EXAM SOLUTIONS

QUESTION 1
(a) Transient states $\{1\}$
(b) Periodic states $\{3, 5\}$
(c) Communicating classes $\{2, 4\}$ and $\{3, 5\}$
(d) $\pi_2 = \pi_4 = 1/2$

QUESTION 2
(a)

$$\begin{aligned} E[M(t)] &= E[N(t) - f(t)] \\ 0 &= E[X(t)^2 + Y(t)^2 + Z(t)^2] - E[f(t)] \\ E[f(t)] &= t + t + t = 3t \\ &\Downarrow \\ f(t) &= 3t \end{aligned}$$

(b)
$E[M(t)] = 0$ is not enough to show that M(t) is a martingale. In order to show it is a martingale, we also need to show

$$E[M(t)|M(t-1), M(t-2), \ldots, M(0)] = M(t-1)$$

QUESTION 3

$$\begin{aligned} P\left\{\underbrace{B(t) \neq 0, a < t \leq b}_{A} \,|\, \underbrace{B(t) \neq 0, b < t \leq c}_{B}\right\} &= \frac{P\left\{\overbrace{B(t) \neq 0, a < t \leq b}^{A} \cap \overbrace{B(t) \neq 0, b < t \leq c}^{B}\right\}}{P\left\{\underbrace{B(t) \neq 0, b < t \leq c}_{B}\right\}} \\ &= \frac{1 - 2/\pi \arctan\sqrt{\frac{c-a}{a}}}{1 - 2/\pi \arctan\sqrt{\frac{c-b}{b}}} \end{aligned}$$

(a)

$$\lim_{c \to b} \frac{1 - 2/\pi \arctan\sqrt{\frac{c-a}{a}}}{1 - 2/\pi \arctan\sqrt{\frac{c-b}{b}}} = 1 - 2/\pi \arctan\sqrt{\frac{b-a}{a}}$$

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since $1 - 2/\pi \arctan(0) = 1$
(b)

$$\lim_{c\to\infty} \frac{1 - 2/\pi \arctan\sqrt{\frac{c-a}{a}}}{1 - 2/\pi \arctan\sqrt{\frac{c-b}{b}}} = \sqrt{\frac{a}{b}}$$

using l'Hopitals rule.

QUESTION 4

(a)
In general,

$$\begin{aligned} P(\max_{0<s<t} B(s) \geq z, B(t) \leq x) &= P(B(t) \geq 2z - x) \\ &= 1 - \Phi\left(\frac{2z - x}{\sqrt{t}}\right) \\ z = \frac{70-60}{20} = .5 \quad &, \quad x = \frac{60-60}{20} = 0, t = 1 \text{ time unit defined as 4 months} \\ P(\text{Make money}) &= 1 - \Phi\left(2(.5) - 0\right) = 1 - \Phi\left(1\right) \end{aligned}$$

Alternatively, consider our path of interest as the reflected Brownian motion at 70. This unreflected path would be above 80 to be like our "above 70 and then back below 60" path. So P(Make money)=1-$\Phi(\frac{80-60}{20}) = 1 - \Phi(1)$ which is the same as before.

QUESTION 5

(a)

$$\begin{aligned} &\text{Case 1: } t \leq 1 \\ &X(t) = N(t) \\ &Y(t) = 0 \\ &\text{Case 2: } t \geq 1 \\ &X(t) = N(t) - N(t-1) \sim \text{Pois}(\lambda) \\ &Y(t) = N(t-1) \sim \text{Pois}((t-1)\lambda) \end{aligned}$$

$X(t)$ and $Y(t)$ are independent.

(b)
This was a typo. The intended question was is X(t) a Poisson variable. The answer is no. Whether or not Y(t) is a Poisson variable is hard to tell so any answer is correct.

QUESTION 6

Assuming $0 \leq s \leq t$

$$\begin{aligned}\text{Cov}[N(t), N(s)] &= E[N(t)N(s)] - \lambda t \lambda s \\ &= E[N(t)N(s)] - \lambda^2 ts\end{aligned}$$

$$\begin{aligned}E[N(t)N(s)] &= E[N(t) - N(s) + N(s)N(s)] = E[N(t) - N(s)N(s)] + E[N(s)^2] \\ E[N(s)^2] &= Var[N(s)] + (E[N(s)])^2 = \lambda s + (\lambda s)^2 \\ E[N(t) - N(s)N(s)] &= E[N(t) - N(s)]E[N(s)] = \lambda(t-s)\lambda s\end{aligned}$$

combining all terms, we have

$$\text{Cov}[N(t), N(s)] = \lambda(t-s)\lambda s + \lambda s + \lambda^2 s^2 - \lambda t \lambda s = \lambda s$$

Similarly for $0 \le t \le s$ we have $\text{Cov}[N(t), N(s)] = \lambda t$, hence

$$\text{Cov}[N(t), N(s)] = \lambda \min(t, s)$$

QUESTION 7

(a)

$$\text{Model 1} \quad \begin{bmatrix} p & \frac{1-p}{6} & \frac{1-p}{6} & \frac{1-p}{6} & \frac{1-p}{6} & \frac{1-p}{6} & \frac{1-p}{6} \\ 1-p & p & 0 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 & 0 & 0 \\ 1-p & 0 & 0 & p & 0 & 0 & 0 \\ 1-p & 0 & 0 & 0 & p & 0 & 0 \\ 1-p & 0 & 0 & 0 & 0 & p & 0 \\ 1-p & 0 & 0 & 0 & 0 & 0 & p \end{bmatrix}$$

$$\text{Model 2} \quad \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

(b) Inspection of Model 2 tells us $\pi_{\text{main}} = 1/2$

(c) Probability of not being in the main room is 1/2 and all six rooms are exchangeable so $1/2 \times 1/6 = 1/12$=P(mink in room A).

(d)

$$\text{Model 1} \quad \begin{bmatrix} p & \frac{1-p}{6} & \frac{1-p}{6} & \frac{1-p}{6} & \frac{1-p}{6} & \frac{1-p}{6} & \frac{1-p}{6} \\ 1-p-\epsilon_1 & p+\epsilon_1 & 0 & 0 & 0 & 0 & 0 \\ 1-p-\epsilon_2 & 0 & p+\epsilon_2 & 0 & 0 & 0 & 0 \\ 1-p & 0 & 0 & p & 0 & 0 & 0 \\ 1-p & 0 & 0 & 0 & p & 0 & 0 \\ 1-p & 0 & 0 & 0 & 0 & p & 0 \\ 1-p & 0 & 0 & 0 & 0 & 0 & p \end{bmatrix}$$