Dean Foster **Stat 433: Final Exam** May 6, 2008

You are allowed both your midterm cheat-sheet and a new one if you made it up. No calculators!

1. Suppose we have a 5 state Markov chain represented by the following matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & .9 & 0 & .1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & .1 & 0 & .9 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

   where we label the states 1-5.

   (a) Which states are transient?

   (b) Which states are periodic?

   (c) What are the communication classes for this process?

   (d) If the process started in second state, what will the distribution be in the limit?

2. Consider 3 standard Brownian motions, $X(t), Y(t)$, and $Z(t)$. Define the norm process as

$$N(t) = X(t)^2 + Y(t)^2 + Z(t)^2.$$

   Define $M(t) = N(t) - f(t)$ for some deterministic function $f(t)$.

   (a) What would the function $f(t)$ have to be to make $E(M(t)) = 0$ for all t?

   (b) For this $f()$, is the fact that $E(M(t)) = 0$ enough to show that $M(t)$ is a martingale?

3. Find the conditional probability that a standard Brownian motion is not zero in the interval $(a, b]$ given that it is not zero in the interval $(b, c]$.

   (a) What is the limit as $c \to b$? Does this make sense?

   (b) What is the limit as $c \to \infty$? Does this make sense?

   (NOTE: in case you forgot to add the arc-tan rule to you cheat sheet it says the probability of a standard Brownian motion having at least one zero between t and $t + s$ is $\frac{2}{\pi} \arctan \sqrt{s/t}$.)

4. I have a scheme for buying and selling "Bush" futures contracts. (Recall the graph I passed out in class.) Currently they are trading for 60 points. My scheme will make me money if over the next 4 months the price goes over 70 points and then ends up at less than 60 points at the end of the 4 months. If this doesn't occur, I will losing money. Assume that the standard deviation over this time period period is 20 points. What is the probability of my scheme making money? (Give me a formula for the answer AND a crude guess as to the numeric value of the probability.)

5. Let $\{N(t); t \geq 0\}$ be a Poisson process of rate λ, representing the arrival process of customers entering a slow food joint. Each person takes exactly one hour to get their food, eat it and leave. Let $X(t)$ denote the number of customers remaining in the store at time t and $Y(t)$ be the number that have come and departed by time t.

   (a) Find the joint distribution of $X(t)$ and $Y(t)$.

   (b) Is $Y(t)$ a Poisson process? Justify your answer.

6. What is the covariance function for a Poisson process? In other words, what is $Cov(N(t), N(s))$ where $N(t)$ is a Poisson process with rate λ.

7. Consider a mink that is put in a cage that has one central room and 6 side rooms. The main room is called M, and the side rooms are called A, B, C, D, E and F. None of the side rooms are connected to each other. In other words, when the mink leaves a side room, he must move back into the main room.

   In each period the mink either stays put with probability p or moves to one of the adjacent rooms with probability 1-p.

   (a) Write down two different probabilistic models for this problem. (One of the should have 7 states, and the other should have only two states).

   (b) Find the probability that the mink is in the main room.

   (c) Find the probability that the mink is in room A.

   (d) Suppose rooms A and B have interesting toys in them. (Say room A has a swimming pool–a favorite of minks, and room B has a traffic cone–which is much less popular.) Hence once the mink enters either room A or room B, there is a much higher probability of staying than in the other side rooms. If possible, modify your models to accommodate this change.

BONUS. Prove or disprove that $M(t)$ in problem 2 is a martingale.